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ROTATIONALLY SYMMETRIC SPONTANEOUS SWIRLING IN MHD FLOWS

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The stability of steady axisymmetric MHD *flows of an inviscid, incompressible, perfectly conducting fluid with respect to swirling — perturbations of the azimuthal components of the velocity field — is studied in a linear approximation. It is shown that for flows similar to a magnetohydrodynamic Hill–Shafranov vortex, the problem reduces to a one-dimensional problem on a closed streamline of the unperturbed flow (the arc length of the streamline is the spatial coordinate). A spectral boundary-value eigenvalue problem is formulated for a system of two ordinary differential equations with periodic coefficients and periodic boundary conditions. Sufficient conditions under which swirling is impossible are obtained. Numerical solution of the characteristic equation shows that, under certain conditions, for each streamline there is a real eigenvalue that yields monotonic exponential growth of the initial perturbations.*

The problem of spontaneous swirling is whether a rotationally symmetric flow can appear in the absence of evident sources of rotation, i.e., when axisymmetric motion without swirl is certainly possible.

Formation of a bathtub vortex may serve as the simplest example [1]. In this case as well as in the case of intense mesoscale atmospheric vortices (such as dust columns, waterspouts, and tornados), the mechanism responsible for generation of swirling motion is not completely understood. It is possible that spontaneous rotation plays an important role in this mechanism. The problem of spontaneous rotation was considered in [2, 3], where examples of approximate solutions describing this phenomenon are given. However, in these examples, a rotating fluid flows into the domain considered, and this makes them insufficiently convincing.

A more rigorous formulation of this problem is given in [4, 5]. The statement of the problem proposed in those papers guarantees a strict control over the kinematic flux of the axial component of the angular momentum, which excludes inflow of the rotating fluid in the flow domain. In the case of MHD flows, in [6] it is shown that it is necessary to control the flux of the axial component of the angular momentum transferred by the magnetic field and the problem is formulated so as to exclude inflow of this component of the angular momentum. The appearance of a swirling flow in a viscous fluid is treated as a bifurcation from the initial axisymmetric flow to a swirling flow due to instability [2], i.e., in this case, for fixed boundary conditions the problem has at least two solutions — with and without swirl. For inviscid flows, this problem is meaningless. In this case, there exist many axisymmetric solutions (without swirl) and rotationally symmetric solutions (with swirl) in a bounded domain. Therefore, the appearance of spontaneous swirling is treated as instability of the initial axisymmetric flow that leads to growth of the amplitude of the azimuthal velocity and an increase in the azimuthal component of the kinetic energy at the expense of the poloidal one (in the exact nonlinear formulation, their sum remains constant because of the law of conservation of energy). To avoid misunderstandings, we point out that the appearance of swirling flow does not violate the law of conservation of angular momentum. In an inviscid fluid, differential rotation which conserves angular

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momentum appears, and in a viscous fluid, subject to the no-slip condition on the boundaries of the flow domain, angular momentum is not necessarily conserved and a swirling flow similar to a bathtub vortex can appear.

The difficulties encountered in studies of three-dimensional flows prompts one to a search for the simplest situations where the phenomenon considered is possible. In this connection, the stability of steady axisymmetric flows with respect to swirling with rotationally symmetric perturbations has been studied.

It has been shown [4, 5] that the bifurcation of axisymmetric flow (appearance of rotationally symmetric flow) does not occur for an arbitrary compressible fluid with variable viscosity. For axisymmetric MHD flows of a viscous incompressible fluid with finite conductivity in a magnetic field, it is shown [6] that rotationally symmetric spontaneous swirling flow is impossible if the section of the flow domain by a meridian plane is simply connected. In such domains, the poloidal components of the magnetic field decay with time because of finite conductivity.

For a perfectly conducting fluid, the character of the connectedness of the flow domain is of no importance because, in axisymmetric flows of such a fluid, the poloidal components of the magnetic field do not vanish since the magnetic field is frozen-in, and, as was shown in [7], under certain conditions, there is instability against swirling (linear growth of azimuthal perturbations with time).

In the present paper, we use a linear approximation to study the possibility of rotationally symmetric spontaneous swirling flow resulting from exponential instability of the initial axisymmetric flow of an inviscid perfectly conducting fluid in a magnetic field of the type of a Hill–Shafranov vortex in a bounded domain.

In the conventional notation, flows of a perfectly conducting incompressible fluid in a magnetic field are described by the following system of equations (fluid density $\rho = 1$):

$$\boldsymbol{v}_t - \boldsymbol{v} \times \boldsymbol{\omega} + \boldsymbol{h} \times \boldsymbol{j} = -\nabla f, \quad f = p + \boldsymbol{v}^2/2, \quad \boldsymbol{j} = \operatorname{rot} \boldsymbol{h}, \quad \boldsymbol{\omega} = \operatorname{rot} \boldsymbol{v},$$

$$\boldsymbol{h}_t = \operatorname{rot}(\boldsymbol{v} \times \boldsymbol{h}), \quad \operatorname{div} \boldsymbol{v} = 0, \quad \operatorname{div} \boldsymbol{h} = 0, \quad \boldsymbol{h} = \boldsymbol{H}/\sqrt{4\pi}.$$

In cylindrical coordinates $\boldsymbol{r} = (r, \varphi, z)$, $\boldsymbol{v} = (u, v, w)$, $\boldsymbol{h} = (h_1, h, h_3)$, the steady rotationally symmetric flows considered are given by the relations

$$u = -\frac{\alpha(\psi)}{r}\frac{\partial \psi}{\partial z}, \quad w = \frac{\alpha(\psi)}{r}\frac{\partial \psi}{\partial r}, \quad h_1 = -\frac{\beta(\psi)}{r}\frac{\partial \psi}{\partial z}, \quad h_3 = \frac{\beta(\psi)}{r}\frac{\partial \psi}{\partial r}, \tag{1}$$

where $\alpha(\psi)$ and $\beta(\psi)$ are arbitrary functions of ψ,

$$v = \alpha\Gamma(\psi)/r + \beta\Omega(\psi)r, \qquad h = \alpha\Omega(\psi)r + \beta\Gamma(\psi)/r, \tag{2}$$

and by the Grad–Shafranov equation for ψ

$$D\psi + \frac{\alpha\alpha' - \beta\beta'}{\alpha^2 - \beta^2}(\psi_r^2 + \psi_z^2) = \frac{r^2}{\alpha^2 - \beta^2} f'(\psi) - \Gamma\Gamma' - \Omega\Omega' r^4, \tag{3}$$

where

$$D = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

In Eqs. (1)–(3), one of the functions (α or β) can be taken equal to 1 without loss of generality. The functions $f(\psi)$, $\Gamma(\psi)$, and $\Omega(\psi)$ can depend on ψ arbitrarily. Since we consider the stability of a steady axisymmetric flow, it is necessary to set $\Gamma = 0$ and $\Omega = 0$ in the initial flows. Next, the functions α and β are chosen to be constant and $\alpha = 1$. Then, β has the meaning of the factor of proportionality between the poloidal components of the velocity and the magnetic field in the initial steady flow: $\boldsymbol{h}_p = \beta \boldsymbol{v}_p$. In this case, the Grad–Shafranov equation simplifies to

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{r^2}{1 - \beta^2} f'(\psi). \tag{4}$$

The pressure is determined by Bernoulli's integral along the streamline $p + (u^2 + v^2 + w^2)/2 = f(\psi)$.

Below we consider flows in a bounded domain and, therefore, there is no need to match the inner and the outer flows. As a result, it becomes possible to obtain a wide class of exact analytical solutions that describe the initial flows.

For $f(\psi) = K_1\psi + K_2\psi^2$ with some constants K_1 and K_2, there exist solutions of the form $\psi(r, z) = g_0(r) + g_2(r)z^2 + \ldots + g_{2n}z^{2n}$ that are symmetric about the plane $z = 0$ (and nonsymmetric if odd powers of z are added) and solutions of the form $\psi(r, z) = g(r)\sin(kz)$ (if $K_1 = 0$). Determination of flows of this form reduces to integration of ordinary differential equations. In particular, among solutions of the first type there is a solution that corresponds to the well-known Hill–Shafranov vortex in a ball generalized to the case of an ellipsoid of revolution. This solution is given by the formula

$$\psi(r, z) = Ur^2(1 - r^2/r_*^2 - z^2r/z_*^2)/2.$$

In these flows, the streamlines are closed and any closed streamline can be treated as the boundary of the flow domain. This allows one to study the stability against rotation for a wide class of flows given by exact analytical solutions in simply and multiply connected domains.

In what follows, we consider the problem of the stability of a steady axisymmetric flow ($v = 0$ and $h = 0$; accordingly, $\Gamma = \Omega = 0$) against swirling — appearance of rotationally symmetric flow ($v \neq 0$). In a linear approximation, the evolution of the azimuthal components of the velocity and the magnetic field is not coupled with the evolution of the poloidal components and can be considered independently. The main aim is to determine whether exponentially growing solutions exist for values of β in the interval $0 < \beta < 1$ and to obtain conditions under which this is possible.

The azimuthal components of the velocity $v_\varphi = v$ and the magnetic field $h_\varphi = h$ satisfy the equations

$$v_t + uv_r + wv_z + \frac{uv}{r} = h_1h_r + h_3h_z + \frac{h_1h}{r}, \quad h_t + uh_r + wh_z - \frac{uh}{r} = h_1v_r + h_3v_z - \frac{h_1v}{r}. \tag{5}$$

In a linear approximation, $u(r, z)$, $w(r, z)$, $h_1(r, z)$, and $h_3(r, z)$ do not depend on time and coincide with their initial values. On the boundary of the axisymmetric domain D, the following conditions must be satisfied: $\boldsymbol{vn} = 0$ and $\boldsymbol{hn} = 0$, where $\boldsymbol{n} = (n_r, 0, n_z)$ is the outward unit normal to the boundary of the flow domain D.

We assume that there is a steady solution of Eq. (4) in an axisymmetric domain D satisfying the boundary conditions formulated above. As shown above, such flows exist. As an example, we take the magnetohydrodynamic Hill–Shafranov vortex and consider the flow inside a sphere on whose boundary $\psi = 0$. In this case, $\psi = r^2(1 - r^2 - z^2)/2$. Here and below, we use the dimensionless variables (lengths are measured in the radii of the sphere, velocities are scaled by the velocity at the center of the sphere, and time is measured in the units of the ratio of the sphere radius to the above-mentioned velocity). Under these conditions, system (5) takes the form

$$\begin{gathered} v_t + u(v - \beta h)_r + w(v - \beta h)_z + u(v - \beta h)/r = 0, \\ h_t + u(h - \beta v)_r + w(h - \beta v)_z - u(h - \beta v)/r = 0, \end{gathered} \tag{6}$$

where, in particular, $u = rz$ and $w = 1 - 2r^2 - z^2$ for the Hill–Shafranov vortex. The structure of Eqs. (6) makes it possible to consider them on any closed streamline. In this case, the value of ψ on the chosen streamline plays the role of a parameter. Indeed, we have

$$u\,\frac{\partial}{\partial r} + w\,\frac{\partial}{\partial z} = q(s)\,\frac{\partial}{\partial s}. \tag{7}$$

Here $q(s) = \sqrt{u^2 + w^2}$ is the absolute value of the velocity [s is the arc length along the streamline reckoned from the point where $r(s)$ attains a maximum value on the streamline]. Since the streamlines are closed, the functions $q(s)$ and $r(s)$ for the initial flow are periodic with period $s_*(\psi)$ determined by the total length of the chosen streamline. We assume that on the streamlines considered, the minimum values $q(0) > 0$ and $r(0) = r_0 > 0$.

In view of (7), Eqs. (6) take the form

$$(rv)_t + q(s)\frac{\partial}{\partial s}r(v-\beta h) = 0, \quad \left(\frac{h}{r}\right)_t + q(s)\frac{\partial}{\partial s}\frac{h-\beta v}{r} = 0. \tag{8}$$

Let $A = (v-\beta h)r/r_0$ and $B = (h-\beta v)r_0/r$. Instead of the variable s, we introduce a variable x such that

$$x = \int_0^s \frac{ds}{q(s)}. \tag{9}$$

For the Hill–Shafranov vortex in a ball, the dependence of $r(x)$ and $z(x)$ along the streamline passing through the point ($r = r_0$, $z = 0$) is obtained from the following system of differential equations:

$$\frac{dr}{dx} = rz, \quad \frac{dz}{dx} = 1 - 2r^2 - z^2; \quad r = r_0 \quad \left(0 < r_0 \leqslant \frac{1}{\sqrt{2}}\right), \quad z = 0 \quad \text{for} \quad x = 0.$$

Eliminating z, for r we obtain the equation $r'' = r - 2r^3$. From this, using the boundary conditions, we obtain $r'^2 = r^2 - r^4 - r_0^2 + r_0^4$ and, finally, we have $\left(\mu = \sqrt{1-r_0^2}\right)$

$$r(x) = r_0/\mathrm{dn}\,(\mu x), \qquad z(x) = r_0\mu k^2\,\mathrm{sn}\,(\mu x)\,\mathrm{cn}\,(\mu x)/\mathrm{dn}\,(\mu x),$$

$$x_* = x(s_*) = 2K(k)/\mu, \quad k = \sqrt{1-2r_0^2}/\mu, \quad r(x+x_*) = r(x), \quad z(x+x_*) = z(x),$$

where $\mathrm{sn}\,(\mu x)$, $\mathrm{cn}\,(\mu x)$, and $\mathrm{dn}\,(\mu x)$ are corresponding Jacobian elliptic functions and $K(k)$ is a complete elliptic integral of the first kind.

As a result, in view of (9), Eqs. (8) become

$$A_t + A_x - \beta g(x)B_x = 0, \qquad B_t + B_x - \beta f(x)A_x = 0, \tag{10}$$

where $g(x) = (r(x)/r_0)^2$ and $f(x) = (r_0/r(x))^2$ $[g(x)f(x) = 1]$. Solving these equations for A_x and B_x, we obtain

$$(1-\beta^2)A_x = -A_t - \beta g(x)B_t, \qquad (1-\beta^2)B_x = -B_t - \beta f(x)A_t. \tag{11}$$

We seek a solution of these equations in the form

$$A(x,t) = a(x)\exp(-\lambda_* t), \qquad B(x,t) = b(x)\exp(-\lambda_* t), \tag{12}$$

where it is convenient to represent λ_* as $\lambda_* = (1-\beta^2)\lambda$. Substituting (12) into (11), we obtain the following system of equations for $a(x)$ and $b(x)$:

$$a' = \lambda a + \beta\lambda g(x)b, \qquad b' = \lambda b + \beta\lambda f(x)a. \tag{13}$$

Solutions of system (13) must be periodic with the period equal to x_*. This requirement determines a point set (as will be shown below) of eigenvalues λ_n for which a nontrivial solution of this system exists. If there is even a single eigenvalue with a negative real part, the initial flow is unstable and swirling flow appears.

Let us give certain *a priori* estimates. We integrate the sum $(\bar{b}a)' + (\bar{a}b)'$ over the period. This yields

$$(\lambda + \bar{\lambda})\int_0^{x_*}(\bar{b}a + \bar{a}b + \beta g(x)|b|^2 + \beta f(x)|a|^2)\,dx = 0. \tag{14}$$

From (14) it follows that λ is purely imaginary and there are no exponentially growing solutions of system (10) if β is equal to or greater than unity. This coincides with the result obtained earlier in [7].

Let $a = rU$ and $b = V/r$. Then, for U and V we obtain the equations

$$U' = (\lambda - z(x))U + \beta\lambda V, \qquad V' = (\lambda + z(x))V + \beta\lambda U,$$

where $z(x) = g'(x)/(2g(x)) = r'(x)/r(x)$ is the value of the coordinate z on the streamline. From this, using the complex conjugate form of these equations, we obtain

$$(\lambda+\bar\lambda)\int\limits_0^{x_*}(|U|^2+|V|^2)\,dx = 2\int\limits_0^{x_*} z(x)(|U|^2-|V|^2)\,dx - \beta(\lambda+\bar\lambda)\int\limits_0^{x_*}(\bar V U+\bar U V)\,dx,$$

$$(\lambda+\bar\lambda)\int\limits_0^{x_*}[(\bar V U+\bar U V)+\beta(|U|^2+|V|^2)]\,dx = 0.$$

Combining these equalities, we find that

$$(\lambda+\bar\lambda)(1-\beta^2)\int\limits_0^{x_*}(|U|^2+|V|^2)\,dx = 2\int\limits_0^{x_*} z(x)(|U|^2-|V|^2)\,dx.$$

Since on each streamline, $|z(x)|\leqslant 1$, it is obvious that $|(\lambda+\bar\lambda)|(1-\beta^2)\leqslant 2$. Thus, the eigenvalues λ_* lie in the strip $|\mathrm{Re}\,\lambda_*|\leqslant 1$.

Similarly, integrating the quantity $aa'+bb'$ over the period, we obtain for real values of λ ($\mathrm{Im}\,\lambda=0$):

$$\lambda\int\limits_0^{x_*}(2\beta(g(x)+f(x))ab+|b|^2+|a|^2)\,dx = 0. \tag{15}$$

It follows from (15) that monotonic rotation is impossible if $\beta(p+1)<1$, where $p=\max(r(x)/r_0)^2$ in the domain considered. For the Hill–Shafranov vortex, we have $p=(1-r_0^2)/r_0^2$.

To determine the eigenvalues in the above-formulated problem, we follow the general theory of linear systems of differential equations with periodic coefficients [8] and obtain the matrizant of system (13) and the corresponding characteristic equation.

System (13) has two linearly independent solutions $(a_1(x), b_1(x))$ and $(a_2(x), b_2(x))$ that assume the values $a_1(0)=1$, $b_1(0)=0$ and $a_2(0)=0$, $b_2(0)=1$ at the point $x=0$ and are obtained by the successive approximation method and can be presented in the form of the converging series

$$a_1(x)=\exp(\lambda x)(1+A_2(x)(\beta\lambda)^2+A_4(x)(\beta\lambda)^4+\ldots),$$

$$b_1(x)=\exp(\lambda x)(B_1(x)(\beta\lambda)+B_3(x)(\beta\lambda)^3+\ldots),$$

$$a_2(x)=\exp(\lambda x)(A_1(x)(\beta\lambda)+A_3(x)(\beta\lambda)^3+\ldots),$$

$$b_2(x)=\exp(\lambda x)(1+B_2(x)(\beta\lambda)^2+B_4(x)(\beta\lambda)^4+\ldots).$$

Here A_n and B_n are defined by the formulas

$$A_1(x)=\int\limits_0^x g(x_1)\,dx_1,\qquad B_1(x)=\int\limits_0^x f(x_1)\,dx_1,$$

$$A_2(x)=\int\limits_0^x f(x_1)\,dx_1\int\limits_0^{x_1} g(x_2)\,dx_2,\qquad B_2(x)=\int\limits_0^x g(x_1)\,dx_1\int\limits_0^{x_1} f(x_2)\,dx_2,$$

$$A_3(x)=\int\limits_0^x g(x_1)\,dx_1\int\limits_0^{x_1} f(x_2)\,dx_2\int\limits_0^{x_2} g(x_3)\,dx_3,$$

$$B_3(x)=\int\limits_0^x f(x_1)\,dx_1\int\limits_0^{x_1} g(x_2)\,dx_2\int\limits_0^{x_2} f(x_3)\,dx_3,$$

$$A_4(x) = \int_0^x f(x_1)\,dx_1 \int_0^{x_1} g(x_2)\,dx_2 \int_0^{x_2} f(x_3)\,dx_3 \int_0^{x_3} g(x_4)\,dx_4,$$

$$B_4(x) = \int_0^x g(x_1)\,dx_1 \int_0^{x_1} f(x_2)\,dx_2 \int_0^{x_2} g(x_3)\,dx_3 \int_0^{x_3} f(x_4)\,dx_4,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The characteristic equation for the multiplicators ρ is given by

$$\rho^2 - (a_1 + b_2)\rho + a_1 b_2 - a_2 b_1 = 0. \tag{16}$$

Equation (16) has two roots ρ_1 and ρ_2, for which the following equalities hold:

$$\rho_1\rho_2 = a_1 b_2 - a_2 b_1 = \exp(2\lambda x_*) \tag{17}$$

(here the second equality is valid by virtue of Liouville's formula) and

$$\rho_1 + \rho_2 = a_1 + b_2. \tag{18}$$

In (16)–(18), the values of the functions $a_1(x)$ and $b_2(x)$ correspond to $x = x_*$.

From the requirement of periodicity it follows that at least one of the multiplicators must be equal to unity. Using this, from (17) and (18), we have

$$1 + \exp(2\lambda x_*) = a_1(x_*) + b_2(x_*). \tag{19}$$

Substituting the functions obtained above into the right side of this equation, we have

$$1 + \exp(2\lambda x_*) = \exp(\lambda x_*)(2 + C_2(\beta\lambda)^2 + C_4(\beta\lambda)^4 + \ldots), \tag{20}$$

where C_{2n} are constants defined by the formulas $C_2 = A_2(x_*) + B_2(x_*) = A_1(x_*)B_1(x_*)$ and $C_{2n} = A_{2n}(x_*) + B_{2n}(x_*)$ $(n \geqslant 2)$. Introducing the notation $\lambda x_* = \zeta$, $C_2 = c$, and $C_{2n} = 2x_*^{2n} c_{2n}/(2n)!$ $(n \geqslant 2)$, we transform Eq. (20) to

$$\cosh\zeta = 1 + c(\beta\zeta)^2/2! + c_4(\beta\zeta)^4/4! + \ldots\ . \tag{21}$$

For the quantities c_{2n}, the following estimates are valid: $p^{-n} < c_{2n} < p^n$ $(n \geqslant 1)$. It follows from these relations that the series on the right side of (21) converges for any $\beta\zeta$ and is an entire function in the complex plane ζ, and the set of eigenvalues is a point set. Numerical calculations for the Hill–Shafranov vortex show that for c_{2n}, stronger inequalities than those presented above hold, namely: $1 < c < p$ and $1 < c_{2n} < c^n$ $(n \geqslant 2)$. The first of these inequalities has been proved rigorously for an arbitrary function $r(x)$, i.e., for arbitrary axisymmetric flows with closed streamlines. However, we did not prove these inequalities rigorously for $n \geqslant 2$ in the general case.

We now expand $\cosh\zeta$ into a series. As a result, after dividing by $(\beta\zeta)^2$ [the eigenvalue $\zeta = 0$ corresponds to a steady swirling flow (2)], we obtain

$$(1 - \beta^2 c)/2! + (1 - \beta^4 c_4)\zeta^2/4! + (1 - \beta^6 c_6)\zeta^4/6! + \ldots + (1 - \beta^{2n} c_{2n})\zeta^{2(n-1)}/(2n)! + \ldots = 0. \tag{22}$$

As was mentioned above, roots of this equation that have negative real parts correspond to exponential growth of the initial perturbations. From (22) it follows that a sufficient condition for the existence of such a root is the existence of a root with a nonzero real part, because if a root ζ exists, the root $-\zeta$ also exists. Using this fact, below we consider only roots that lie in the right half-plane $\mathrm{Re}\,\zeta \geqslant 0$. Attempts to seek a root in the form of an expansion in powers of β do not lead to the answer since a root with nonzero real part can appear only for finite values of the quantities $\beta^{2n} C_{2n}$. For $\beta = 0$, the eigenvalues are equal to $\zeta_n = 2\pi n i$. For $\beta \ll 1$, assuming that the function $\zeta_n(\beta)$ is analytic, we can seek roots in the form of power series in β, but in this case only purely imaginary quantities are obtained. The series thus obtained have finite radius of convergence, beyond which branch points and roots with nonzero real part can appear.

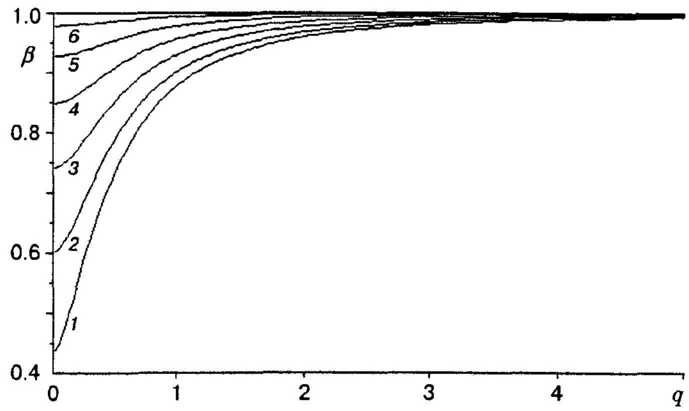

Fig. 1

To find real roots and determine the range of values of β for which they can appear, we used the following numerical procedure. (These calculations were performed together with M. S. Kotel'nikova.) From Eq. (19), we obtained the function $\beta(q)$ $(q = \beta\zeta)$:

$$1 + \exp(2q/\beta) = \exp(q/\beta)(\tilde{a}_1(x_*) + \tilde{b}_2(x_*)) = 2\exp(q/\beta)\, d,$$
$$d = (\tilde{a}_1(x_*) + \tilde{b}_2(x_*))/2, \qquad \beta = q/\ln(d + \sqrt{d^2 - 1}), \tag{23}$$

where $\tilde{a}_1(x_*) = a_1(x_*)\exp(-q/\beta)$ and $\tilde{b}_2(x_*) = b_2(x_*)\exp(-q/\beta)$ are the values of solutions at the point x_* for the equations

$$\tilde{a}_1' = qg(x)\tilde{b}_1, \quad \tilde{b}_1' = qf(x)\tilde{a}_1, \qquad \tilde{a}_2' = qg(x)\tilde{b}_2, \quad \tilde{b}_2' = qf(x)\tilde{a}_2 \tag{24}$$

with the initial conditions $\tilde{a}_1(0) = 1$, $\tilde{b}_1(0) = 0$, $\tilde{a}_2(0) = 0$, and $\tilde{b}_2(0) = 1$. Equations (24) were solved numerically. The function $\beta(q)$ was determined from (23). Plots of these functions for various streamlines specified by the quantity r_0 for the Hill–Shafranov vortex are shown in Fig. 1 (curves 1–6 refer to values $r_0 = 0.1$, 0.2, 0.3, 0.4, 0.5, and 0.6, respectively). For small and large values of $|q|$, we can obtain analytic representations of these curves. As $|q| \to 0$, from (23) we obtain

$$\beta = (1 + (c^2 - c_4)q^2/(24c) + O(q^4))/\sqrt{c}.$$

In the case of the Hill–Shafranov vortex, we have $c_4 < c^2$ and for $1 > \beta > 1/\sqrt{c}$ $(c \geqslant 1)$ there is a positive real root (and, therefore, as was shown above, a negative real root) of the characteristic equation that indicates the appearance of instability.

According to the general theory [9], we have the following asymptotic representation of solutions of system (13) for $|q| \to \infty$: $a(x) = rU_*(x)\exp(\lambda x)$ and $b(x) = V_*(x)\exp(\lambda x)/r$, where U_* and V_* satisfy the equations

$$U_*' = \beta\lambda V_* - z(x)U_*, \qquad V_*' = \beta\lambda U_* + z(x)V_*. \tag{25}$$

Let (U_1, V_1) and (U_2, V_2) be linearly independent solutions of this system in the form

$$U_1 = \exp(\beta\lambda x)\sum_{n=0}^{\infty} U_{1,n}(\beta\lambda)^{-n}, \qquad V_1 = \exp(\beta\lambda x)\sum_{n=0}^{\infty} V_{1,n}(\beta\lambda)^{-n},$$

$$U_2 = \exp(-\beta\lambda x)\sum_{n=0}^{\infty} U_{2,n}(\beta\lambda)^{-n}, \qquad V_2 = \exp(-\beta\lambda x)\sum_{n=0}^{\infty} V_{2,n}(\beta\lambda)^{-n},$$

where $U_{1,n}$, $V_{1,n}$, $U_{2,n}$, and $V_{2,n}$ are determined from a recurrent system of equations that is obtained after substitution of these expansions into (25). As a result, we obtain

$$U_{1,0} = 1, \quad U_{1,n+1} = \frac{1}{2}\left(V'_{1,n} - z(x)V_{1,n} - \int\limits_0^{x_*} z(x)(V'_{1,n} - z(x)V_{1,n})\,dx\right),$$

$$V_{1,0} = 1, \quad V_{1,n+1} = -\frac{1}{2}\left(V'_{1,n} - z(x)V_{1,n} + \int\limits_0^{x_*} z(x)(V'_{1,n} - z(x)V_{1,n})\,dx\right),$$

$$U_{2,0} = -1, \quad U_{2,n+1} = \frac{1}{2}\left(V'_{2,n} - z(x)V_{2,n} - \int\limits_0^{x_*} z(x)(V'_{2,n} - z(x)V_{2,n})\,dx\right),$$

$$V_{2,0} = 1, \quad V_{2,n+1} = \frac{1}{2}\left(V'_{2,n} - z(x)V_{2,n} + \int\limits_0^{x_*} z(x)(V'_{1,n} - z(x)V_{2,n})\,dx\right).$$

Using these expansions and neglecting terms of order $O(1/|q|^2)$, we find that $1 + \exp(2q/\beta) = \exp(q/\beta)[(1 + O(1/|q|^2)\cosh q + (\gamma/q + O(1/|q|^2)\sinh q]$ and, for real $q > 0$, we obtain

$$\beta = \frac{1}{1 + \gamma/q^2}, \qquad q = \sqrt{\frac{\beta\gamma}{1-\beta}}, \quad \gamma = \frac{x_*}{2}\int\limits_0^{x_*} z^2(x)\,dx.$$

We recall that the increment of growing perturbations $\lambda_* = (1-\beta^2)\lambda$ and it goes to zero as $\beta \to 1$.

In Fig. 1, one can see that for a specified value of β, λ_* assumes different values on different streamlines (obviously, this is also true for complex eigenvalues). Therefore, there are no solutions of the form $v = \exp(-\lambda_* t)V(r,z)$ and $h = \exp(-\lambda_* t)H(r,z)$, where $\lambda_* \neq 0$ is a constant that is the same for the entire domain. An exception is the value $\lambda_* = 0$, for which a solution exists and corresponds to a steady swirling flow (2).

To determine complex roots, we solved Eq. (22) numerically. The coefficients c_{2n} were calculated using the formula

$$c_{2n} = \frac{(2n)!}{2x_*^{2n}}(A_{2n}(x_*) + B_{2n}(x_*)),$$

where $A_{2n}(x)$ and $A_{2n}(x)$ are solutions of the equations

$$A'_1 = g(x), \quad A'_2 = A_1/g(x), \quad A'_3 = g(x)A_2, \ldots,$$

$$B'_1 = 1/g(x), \quad B'_2 = g(x)B_1, \quad B'_3 = B_2/g(x), \ldots$$

with zero boundary conditions $A_i(0) = B_i(0) = 0$.

As a result, for small absolute values of β, the roots and the range of values of ζ for which the roots appear obtained by these two methods coincided with accuracy up to a third significant digit. In addition, calculations using the second method yielded complex values with nonzero real and imaginary parts, and this indicates that oscillatory instability is possible.

Thus, it has been shown by the numerical calculations that for any streamline ($r_0 > 0$) there is a value of β, namely,

$$1 > \beta > \frac{1}{\sqrt{c}}, \qquad c = \int\limits_0^{x_*} r^2(x)\,dx \int\limits_0^{x_*} \frac{dx}{r^2(x)} \geqslant 1, \tag{26}$$

for which a real eigenvalue exists and, hence, initial perturbations grow exponentially. The results, obtained in the numerical calculation of roots from Eq. (22), suggest that criterion (26) also defines the boundary of nonmonotonic (oscillatory) instability.

Direct numerical calculations of unsteady solutions of Eqs. (10) with the periodic initial data $A = v_0(x)r/r_0$ and $B = -\beta v_0(x)r_0/r$ and periodic boundary conditions [initial data of this form correspond to the azimuthal perturbations $v = v_0(x)$ and $h = 0$] confirm the occurrence of an instability of exponential type and are in agreement with criterion (26).

It has been shown [5–7] that a mechanism that provides for counter-gradient flux of the axial component of angular momentum is necessary for the appearance of a spontaneous swirling flow. The results obtained here indicate the possibility of appearance of rotationally symmetric spontaneous swirling flow due to such a flux related to a magnetic field, at least, in a linear approximation for the model of an inviscid perfectly conducting fluid. The question that remains open is whether this result holds if nonlinearity and viscosity are taken into account.

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REFERENCES

1. M. A. Lavrent'ev and V. V. Shabat, *Problems of Hydrodynamics and Their Mathematical Models* [in Russian], Nauka, Moscow (1973).
2. M. A. Gol'dshtik, E. M. Zhdanova, and V. N. Shtern, "Spontaneous swirling of submerged jet," *Dokl. Akad. Nauk SSSR*, **277**, No. 4, 815–818 (1984).
3. A. M. Sagalakov and A. Yu. Yudintsev, "Three-dimensional self-oscillating magnetohydrodynamic flows of a fluid with finite conductivity in longitudinal magnetic field in an annular channel," *Magn. Gidrodin.*, No. 1, 41–48 (1993).
4. B. A. Lugovtsov, "Is spontaneous swirling of axisymmetric flow possible?" *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 2, 50–54 (1994).
5. B. A. Lugovtsov and Yu. G. Gubarev, "On spontaneous swirling in axisymmetric flows," *Prikl. Mekh. Tekh. Fiz.*, **36**, No. 4, 52–59 (1995).
6. B. A. Lugovtsov, "Spontaneous swirling in axisymmetric flows of a conducting fluid in a magnetic field," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 6, 35–43 (1996).
7. B. A. Lugovtsov, "Spontaneous axisymmetric swirling in an ideally conducting fluid in a magnetic field," *Prikl. Mekh. Tekh. Fiz.*, **38**, No. 6, 29–31 (1997).
8. V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients* [in Russian], Nauka, Moscow (1972).
9. M. V. Fedoryuk, *Asymptotic Methods for Linear Ordinary Differential Equations* [in Russian], Nauka, Moscow (1983).